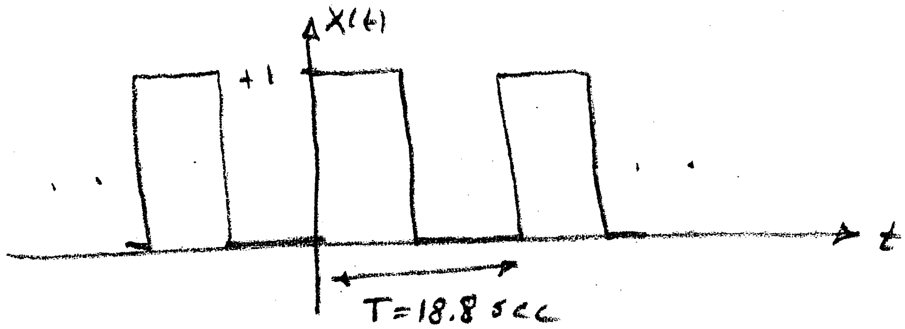
Homework Solutions

S 5)

i) $\omega_o = \frac{\omega_n}{3} = \frac{1}{3} = .333$ rad/sec

$$T = \frac{2\pi}{\omega_o} = \frac{2 \times 3.14}{.333} = 18.8 \text{ sec}$$

ii) The square wave input is



In class we determined the Fourier coefficients for such a square wave as—

$$a_0 = \frac{1}{2}$$

$$a_k = \begin{cases} \frac{1}{k\pi j} & k \text{ odd} \\ 0 & k \text{ even}, k \neq 0 \end{cases}$$

So the Fourier coefficients for $k = 0, \pm 1, \pm 2, \pm 3$ are

$a_0 = \frac{1}{2}$ $\qquad\qquad$ $a_0 = 0.5$

$a_1 = \frac{1}{\pi j}$ $\quad a_{-1} = -\frac{1}{\pi j}$ $\quad \Rightarrow \quad$ $a_1 = \frac{.318}{j}$ $\quad a_{-1} = -\frac{.318}{j}$

$a_2 = a_{-2} = 0$ $\qquad\qquad$ $a_2 = a_{-2} = 0$

$a_3 = \frac{1}{3\pi j}$ $\quad a_{-3} = -\frac{1}{3\pi j}$ $\qquad$ $a_3 = \frac{.106}{j}$ $\quad a_{-3} = -\frac{.106}{j}$

iii) The output Fourier coefficients are

$$b_k = H(jk\omega_0)\, a_k \Rightarrow |b_k| = |H(jk\omega_0)| \cdot |a_k|$$
$$\measuredangle b_k = \measuredangle H(jk\omega_0) + \measuredangle a_k$$

The magnitude and angle of $H(j\omega)$ are

$$|H(j\omega)| = \frac{1}{\left[\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2\right]^{\frac{1}{2}}}$$

$$\measuredangle H(j\omega) = -\tan^{-1}\left[\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]$$

We need $|H(jk\omega_0)|$ and $\measuredangle H(jk\omega_0)$ also $|a_k|$ and $\measuredangle a_k$

k	$k\omega_0$	$\frac{k\omega_0}{\omega_n}$	$\lvert H\rvert$	$\measuredangle H$
0	0	0	1.0	0
1	.333	.333	.996	$-27.7°$
-1	-.333	-.333	.996	$27.7°$
3	1.0	1.0	.715	$-90°$
-3	-1.0	-1.0	.715	$90°$

k	$\lvert a_k\rvert$	$\measuredangle a_k$
0	.5	0
1	.318	$-90°$
-1	.318	$+90°$
3	.106	$-90°$
-3	.106	$90°$

So the b_k's are $b_k = H(jk\omega_0) \cdot a_k$

k	$\lvert b_k\rvert$	$\measuredangle b_k$
0	.50	0
1	.317	$-117.7°$
-1	.317	$+117.7°$
3	.076	$-180°$
-3	.076	$+180°$

← Note that $b_k^* = b_{-k}$ assuming that the output is real!

56) i)

$$|kH(j\omega)| = \frac{k}{\left[\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2\right]^{\frac{1}{2}}}$$

At $\omega = \omega_n = 1.0$ the requirement is

$|kH(j\omega)| = 0.0\,dB \equiv 1.0$

$$\therefore \quad 1 = \frac{k}{\left[(1-1)^2 + (2\zeta)^2\right]^{\frac{1}{2}}} = \frac{k}{2\zeta} \Rightarrow \boxed{k = 2\zeta}$$

At $\omega = 0.1 = \frac{\omega_n}{10}$ the requirement is

$|kH(j\omega)| = -40\,dB \equiv 0.01$

$$\therefore \quad .01 = \frac{k}{\left[\left(1-(.1)^2\right)^2 + \left(2\zeta(.1)\right)^2\right]^{\frac{1}{2}}}$$

substituting from above

$$.01 = \frac{k}{\left[(.99)^2 + (.1k)^2\right]^{\frac{1}{2}}} = \frac{k}{\left[.98 + .01k^2\right]^{\frac{1}{2}}}$$

$\therefore k = .0099 \qquad \zeta = \frac{k}{2} = .0049$

S6) ii)

$$|K\,H(10j)| = \frac{0.0099}{\left[\left(1-(10)^2\right)^2 + \left(0.0099\times 10\right)^2\right]^{1/2}}$$

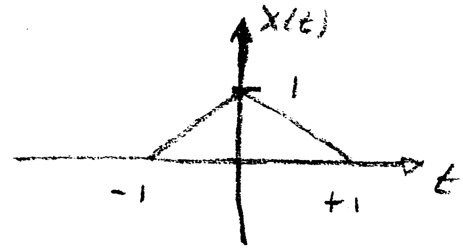
$$= \frac{0.0099}{\left[(-99)^2 + (0.099)^2\right]^{1/2}}$$

$$\cong 1.0\times 10^{-4}$$

$$= -80\ dB$$

So the filter passes signals with frequency content at 1.0 rad/sec and attenuates signals with frequency content at 0.1 rad/sec and 10 rad/sec by two orders ~~of magnitude,~~ and 4 orders of magnitude respectively.

$$x(t) = \begin{cases} t+1 & -1<t<0 \\ 1-t & 0<t<1 \\ 0 & \text{elsewhere} \end{cases}$$



$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{j\omega t} dt = \int_{-\infty}^{\infty} x(t)(\cos\omega t + j\sin\omega t)\,dt$$

We can simplify this integration by observing that

$x(t) \equiv$ even function of t, $x(t) = x(-t)$

$\cos(\omega t) =$ even function of t, $\cos(\omega t) = \cos(-\omega t)$

$\sin(\omega t) =$ odd function of t, $\sin(\omega t) = -\sin(-\omega t)$

Thus

$x(t)\sin(\omega t) \equiv$ odd function of time so its integral from $-\infty$ to $+\infty$ must be zero

$x(t)\cos(\omega) =$ even function of time so

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\cos(\omega t)\,dt = 2\int_{0}^{\infty} x(t)\cos(\omega t)\,dt = 2\int_{0}^{1}(1-t)\cos(\omega t)\,dt$$

$$= 2\int_0^1 \cos\omega t\,dt - 2\int_0^1 t\cos(\omega t)\,dt$$

$$= \frac{2\sin(\omega t)}{\omega}\Big|_0^1 - 2\left[\frac{1}{\omega^2}\cos(\omega t) + \frac{1}{\omega}t\sin(\omega t)\right]_0^1$$

$$= \cancel{\frac{2\sin(\omega)}{\omega}} - \frac{2}{\omega^2}[\cos(\omega) - 1] - \cancel{\frac{2}{\omega}\sin(\omega)} = \frac{2}{\omega^2}[1-\cos(\omega)] = \frac{\sin^2(\frac{\omega}{2})}{(\frac{\omega}{2})^2}$$